Tangles, Trees and Flowers

Ben Clark and Geoff Whittle

Connectivity Functions

A *connectivity function* on a set E is a function λ on the subsets of E such that

- $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$, and
- $\lambda(X) + \lambda(Y) \geq \lambda(X \cup Y) + \lambda(X \cap Y)$ for all $X, Y \subseteq E$.

A partition $(X, E - X)$ is a *k-separation* if $\lambda(X) \leq k$.

The connectivity function of a matroid

Let M be a matroid with rank function r. Let

$$\lambda_M(X) = r(X) + r(E(M) - X) - r(M) + 1.$$

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The connectivity function of a graph

Let G be a graph with edge set E. For $X \subseteq E$, let $\lambda_G(X)$ be the number of vertices that are incident with an edge in both X and $E - X$. Then λ_G is a connectivity function.

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Theorem (Oxley, Semple, W.)

Let M be a 3-connected matroid with at least 9 elements. Then there is a labelled tree T such that, up to equivalence every non-sequential 3-separation is displayed by T.

Theorem (Aikin and Oxley 2011)

Let M be a 4-connected matroid with at least 17 elements. Then there is a labelled tree T such that, up to equivalence every non-sequential 4-separation is displayed by T.

Tangles

A *tangle of order k* in a connectivity function λ is a collection τ of subsets of E such that:

(T1) $\lambda(A) < k$ for all $A \subseteq E$.

(T2) If (A, B) is a $k-1$-separation, then τ contains A or B.

(T3) If $A, B, C \in \tau$, then $A \cup B \cup C \neq E$.

(T4) $E - \{e\} \notin \tau$ for all $e \in E$.

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(T4) $E - \{e\} \notin \tau$ for all $e \in E$.

3- and 4-connected matroids and graphs have unique tangles of order 3 and 4 respectively.

Theorem

Let τ be a robust tangle in a connectivity function of order k. Then there exists a tree T, associated with τ, having the property that every τ-non-sequential k-separation is τ-equivalent to a k-separation displayed by T.

Robust Tangles

The tangle τ is *robust* if whenever $A_1, A_2, \ldots, A_8 \in \tau$, then $A_1 \cup A_2 \cup \cdots \cup A_8 \neq E$.

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- Robustness is just a strengthening of (T3).
- Is robustness needed for graph tangles?

Structure Relative to a Tangle

- A subset X of E is τ-*strong* if it is not contained in a member of τ. Otherwise it is τ-weak.
- We are interested in the τ-strong k-separations.

Full Closure

- A τ-strong k-separating set X is *fully closed* with respect to τ if $\lambda(X \cup Y) > k$ for every non-empty τ-weak set $Y \subseteq E - X$.
- The intersection of all fully-closed k-separating sets that contain X, which we denote by $\mathrm{fcl}_\tau(\mathrm{X})$, is the *full closure* of X with respect to τ.

Sequential k-separations

- A τ-strong k-separation (X, Y) is *sequential* if either $\mathrm{fcl}_\tau(\mathrm{X}) = \mathrm{E}$ or $\mathrm{fcl}_\tau(\mathrm{Y}) = \mathrm{E}$.
- We make no attempt to display sequential k-separations.

Equivalent k-separations

- Let (X, Y) and (X', Y') be τ-strong k-separations. Then (X, Y) and (X', Y') are τ*-equivalent* if

$$\{\mathrm{fcl}_\tau(\mathrm{X}), \mathrm{fcl}_\tau(\mathrm{Y})\} = \{\mathrm{fcl}_\tau(\mathrm{X}'), \mathrm{fcl}_\tau(\mathrm{Y}')\}.$$

- We display non-sequential k-separations up to this equivalence.

Future Work

- Connection with work of Carmesin, Diestel, Hundertmark and Stein.

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- Is there a coherent structure theory for connectivity functions?